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Nontrivial Flavor Structure from Noncompact Lie Group in Noncommutative Geometry

Masaki J.S. Yang

Institute of Particle and Nuclear Studies,
High Energy Accelerator Research Organization (KEK)
Tsukuba 305-0801, Japan

Abstract

In this paper, we propose a mechanism which induces nontrivial flavor structure from transformations of a noncompact Lie group SL(3,C) in noncommutative geometry. Matrices $L \in$ SL(3,C) are associated with the preon fields as $a_{L,R}(x) \to L_{L,R} a_{L,R}(x)$. In order to retain the Hermiticity of the Lagrangian, we assume the same trick when $\psi^\dagger \psi$ is replaced by $\bar{\psi}\psi$ to construct a Lorentz invariant Lagrangian. As a result, the Dirac Lagrangian has both of flavor-universal gauge interactions and nontrivial Yukawa interactions. Removing the unphysical unitary transformations, Yukawa matrices found to be $Y_{ij} = L_L^\dagger k L_R \to \Lambda_L U_L^\dagger k U_R \Lambda_R$. Here, k is a coefficient, U is 3×3 unitary matrix and Λ is the eigenvalue matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \lambda_2 \lambda_3 = 1$. If $L_{L,R}$ are originated from a broken symmetry, the hierarchy and mixing of flavor can be interpreted as the "Lorentz boost" and the "rotation" in this space respectively.

1 Introduction

Although the Higgs boson was found at the LHC [1, 2], the existence of the particle sheds further conundrums, *e.g.,* its theoretical origin, the hierarchy problem, and the flavor puzzle. Among various theories which try to clarify the origin of the Higgs boson, the Yang–Mills–Higgs model in noncommutative geometry (NCG) [3] is an elegant possibility. In this model, the Higgs boson is identified the gauge boson of the fifth dimension that has the noncommutative differential algebra. In this context of NCG, usually the nontrivial flavor structures are introduced by hand to the distance of the extra dimension, $M \to M \otimes K_{ij}$, $K_{ij} = (Y_u, Y_d, Y_e)_{ij}$ in proper representation spaces [4, 5]. A lot of paper treats the intricate flavor structures in the Standard Model [6] as one of the "principles" or "axioms". Meanwhile, in the phenomenological region, innumerable theories and models has been proposed to explain the flavor structures. For example, continuous or discrete, hundreds of flavor symmetries [7], the flavor textures [8], an empirical mass relation [9], and so on.

In this paper, we propose a mechanism which induces nontrivial flavor structure from transformations of a noncompact Lie group SL(3,C) in NCG. Matrices $L \in$ SL(3,C) are associated with the preon fields as $a_{L,R}(x) \to L_{L,R} a_{L,R}(x)$. In order to retain the Hermiticity of the Lagrangian, we assume the same trick when $\psi^\dagger \psi$ is replaced by $\bar{\psi}\psi$ to construct a Lorentz invariant Lagrangian. As a result, the Dirac Lagrangian has both of flavor-universal gauge interactions and nontrivial Yukawa interactions. Removing the unphysical unitary transformations, Yukawa matrices found to be $Y_{ij} = L_L^\dagger k L_R \to \Lambda_L U_L^\dagger k U_R \Lambda_R$. Here, k is a coefficient, U is 3×3 unitary matrix and Λ is the eigenvalue matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \lambda_2 \lambda_3 = 1$. The Yukawa matrices in the real world are indeed more intricate, slightly different to the color and isospin of the each fermions. However, introducing the charge dependent $L_{L,R}$, we can induce more realistic flavor structures.

It is intriguing that the Yukawa matrix is a restricted class of the seesaw like formula of the partial compositeness [10] in the composite Higgs model [11]: $y_{u,d} = \lambda_q M_Q^{-1} Y^{U,D} M_{U,D}^{-1} \lambda_{u,d}$, where $\lambda_{q,u,d}, Y^{U,D}$ are arbitrary matrices, $M_{Q,U}$ are diagonal matrices. Then, we can use the same formulation to the composite Higgs model in analyzing the Yukawa matrices. Inversely, it might be possible to divert the idea of SL(3,C) transformation to the partial compositeness.

If the matrices $L_{L,R}$ are originated from a broken internal symmetry SL(3,C)$_L\times$ SL(3,C)$_R$, the matrices L correspond to the "Lorentz transformation" of the internal symmetry, and then, the hierarchy and mixing of flavor are interpreted as the "Lorentz boost" and the "rotation" respectively.

This paper is organized as follows. In the next section, we review the extended Dirac Lagrangian of the generalized gauge theory in NCG. In Sect. 3, the mechanism induces nontrivial flavor structure is presented. Section 4 is devoted to conclusions.

2 The extended connection and the Dirac Lagrangian

At the beginning, we briefly review the Higgs mechanism in NCG. The following discussions are only confined in the fermionic sector, and those of bosonic sector are found in the review [12]. The spacetime is $M^4 \times Z_2$, taken as the product of the usual Minkowski space and the two discrete points. The coordinates are represented $x^M = (x^\mu, y = \pm)$. Then it leads to the relation $y^2 = 1,\ ydy = -dyy$, which leads to the spontaneous symmetry-breaking mechanism. In particular, we rename the five-dimension as $y = (+,-) \equiv (L, R)$. The extension to the $M^4 \times Z_N$ is straightforward.

The preon field a^i is given as a diagonal matrix [13]

$$a^i = \begin{pmatrix} a^i(x,L) & 0 \\ 0 & a^i(x,R) \end{pmatrix} \equiv \begin{pmatrix} a^i_L(x) & 0 \\ 0 & a^i_R(x) \end{pmatrix}, \tag{1}$$

where each $a^i_{L,R}$ are also matrix-valued functions. The exterior derivative is defined by the commutator

$$\boldsymbol{d}a^i = [D, a^i], \quad D = \begin{pmatrix} d & iM \\ -iM^\dagger & d \end{pmatrix}, \tag{2}$$

where $d = \partial_\mu dx^\mu$. The matrix M determines the pattern of the symmetry breaking. We take the M as a pure imaginal values, as in the old matrix formulations [14], in order to retain the γ^5 is real. The extended connection is introduced as in Ref. [13]:

$$\boldsymbol{A}(x) = \sum_i a^{i\dagger}\boldsymbol{d}a^i = \sum_i a^{i\dagger}[D, a^i] = \begin{pmatrix} A_L & i\Phi \\ -i\Phi^\dagger & A_R \end{pmatrix}, \tag{3}$$

where the index of sum i is assumed to be finite. In the components,

$$A_{(L,R)} = \sum_i a^{i\dagger}_{(L,R)} da^i_{(L,R)}, \quad \Phi = \sum_i a^{i\dagger}_L M a^i_R - M, \tag{4}$$

which satisfies the anti-Hermite condition $\boldsymbol{A}^\dagger = -\boldsymbol{A},\ A^\dagger_{L,R} = -A_{L,R}$. We abbreviate the index i hereafter.

The chiral fermions are assigned on each points as

$$\Psi = \begin{pmatrix} \psi(x,L) \\ \psi(x,R) \end{pmatrix}, \quad \psi(x,L) = \psi_L, \quad \psi(x,R) = \psi_R. \tag{5}$$

In order to describe the Dirac Lagrangian, we introduce the covariant derivative for fermions

$$\boldsymbol{D}_M = \boldsymbol{d}_M + \boldsymbol{A}_M = \left(\begin{pmatrix} \partial_\mu + A_{L\mu} & 0 \\ 0 & \partial_\mu + A_{R\mu} \end{pmatrix}, i\begin{pmatrix} 0 & M+\Phi \\ -M-\Phi^\dagger & 0 \end{pmatrix} \right). \tag{6}$$

Rescaling the connection $\boldsymbol{A}_M \to g\boldsymbol{A}_M$, the Dirac Lagrangian is given by

$$\mathcal{L}_D = \bar{\Psi} i\Gamma^M \boldsymbol{D}_M \Psi = \begin{pmatrix} \bar{\psi}_L & \bar{\psi}_R \end{pmatrix} \begin{pmatrix} i\not{D}_L & -i\gamma^5 H \\ +i\gamma^5 H & i\not{D}_R \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{7}$$

$$= \begin{pmatrix} \bar{\psi}_L & \bar{\psi}_R \end{pmatrix} \left[\begin{pmatrix} i\gamma^\mu \partial_\mu & M \\ M & i\gamma^\mu \partial_\mu \end{pmatrix} + g \begin{pmatrix} \gamma^\mu A_{L\mu} & \Phi \\ \Phi & \gamma^\mu A_{R\mu} \end{pmatrix} \right] \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{8}$$

where $\Gamma^M = (\gamma^\mu, i\gamma^5)$ that satisfies the Clifford algebra $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$. Redefining the four-dimensional connection $A_{L,R} \to -iA_{L,R}$ (because they are anti-Hermite), the covariant derivative are $\not{D}_{(L,R)} = \gamma^\mu(\partial_\mu - igA_{(L,R)\mu})$. $H \equiv M + g\Phi$ is regarded as a physical Higgs field with nonzero vacuum expectation value. In the last line the $i\gamma^5$ is removed by a proper chiral transformation. The field Φ is also normalized in order to obtain the canonical kinetic term $(D_\mu H)^\dagger D^\mu H \in F^{MN} F_{MN}$. Note that the vector space in Eqs. (7),(8) is not the space of the Dirac matrices but discrete Z_2 points in $M^4 \times Z_2$.

3 Nontrivial flavor structure from noncompact Lie group

Next, we propose a mechanism which induces nontrivial flavor structure from transformations of a noncompact Lie group SL(3,C) in NCG. The spacetime is enlarged to $M^4 \times Z_2$ with a three-dimensional internal flavor space. The derivative matrix D is further extended to the flavor space as follows

$$d \to d \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M \to M \otimes \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \equiv M \otimes k, \tag{9}$$

where k is a real number coefficient.

In this flavor space, the preon fields is assumed to be proportional to the identity matrix

$$a_{(L,R)} = \begin{pmatrix} a_{(L,R)}(x) & 0 & 0 \\ 0 & a_{(L,R)}(x) & 0 \\ 0 & 0 & a_{(L,R)}(x) \end{pmatrix}. \tag{10}$$

For later convenience, the conjugation of the preon is renamed as [1]

$$a^\dagger_{L,R}(x) \to \bar{a}_{L,R}(x). \tag{11}$$

Moreover, we assume that internal SL(3,C) matrices are associated with the preon fields as follows:

$$a_{(L,R)}(x) \to L_{(L,R)} a_{(L,R)}(x), \tag{12}$$

$$\bar{a}_{(L,R)}(x) \to \bar{a}_{(L,R)}(x) L^{-1}_{(L,R)}. \tag{13}$$

[1] Indeed, $a^\dagger(x)$ is not necessary to be a real Hermite conjugate of the $a(x)$. The original paper notate this field $b(x)$ [4].

Here, $L_{(L,R)}$ are x independent elements of the internal SL(3,C)

$$L_{ij} = \exp[i\lambda^a_{ij}\alpha^a + \lambda^{\hat{a}}_{ij}\alpha^{\hat{a}}], \tag{14}$$

where $\lambda^a, \lambda^{\hat{a}}$ are the Gell-Mann matrices with $a, \hat{a} = 1 - 8$. By these assumptions, the gauge and Higgs fields are found to be

$$\begin{aligned} A_{(L,R)} &= \sum \bar{a}_{(L,R)} da_{(L,R)} \to \sum \bar{a}_{(L,R)} L^{-1}_{(L,R)} L_{(L,R)} da_{(L,R)} = A_{(L,R)}\delta_{ij}, \\ H &= \sum \bar{a}_L M k a_R \to \sum \bar{a}_L L_L^{-1} M k L_R a_R \to H L^{-1}_{Lik} k L_{Rkj}, & (15) \\ H^\dagger &= \sum \bar{a}_R M k a_L \to \sum a_R^\dagger L_R^{-1} M k L_L a_L \to H^\dagger L^{-1}_{Rik} k L_{Lkj}. & (16) \end{aligned}$$

However, since $L_{(L,R)}$ are not unitary, $[L_L^{-1} L_R]^\dagger \neq L_R^{-1} L_L$ and it induces non-Hermite Lagrangian. Even there might be several method to remedy this point, tentatively we further expand the representation space as follows:

$$\lambda^a \to \begin{pmatrix} \lambda^a & 0 \\ 0 & \lambda^a \end{pmatrix}, \quad -i\lambda^{\hat{a}} \to \begin{pmatrix} -i\lambda^{\hat{a}} & 0 \\ 0 & i\lambda^{\hat{a}} \end{pmatrix}, \quad k \to \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \tag{17}$$

and all other entities are extended trivially. Then, the (anti-)Hermite generators (anti-)commute with k, and then it leads to $L_L^{-1} k L_R = k L_L^\dagger L_R$. This trick is the same when $\psi^\dagger\psi$ is replaced by $\bar{\psi}\psi$ in order to construct a Lorentz invariant Lagrangian.

Accordingly, the Lagrangian is described as

$$\mathcal{L}_D = \bar{\Psi} i \Gamma^M \boldsymbol{D}_M \Psi = \begin{pmatrix} \bar{\psi}_{Li} & \bar{\psi}_{Ri} \end{pmatrix} \begin{pmatrix} i\not{D}_L \otimes 1 & H \otimes k \\ H^\dagger \otimes k & i\not{D}_R \otimes 1 \end{pmatrix} \begin{pmatrix} \psi_{Li} \\ \psi_{Ri} \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} \bar{\psi}_{Li} & \bar{\psi}_{Ri} \end{pmatrix} \begin{pmatrix} i\gamma^\mu(\partial_\mu + igA_{L\mu}) \otimes 1_{ij} & H \otimes Y_{ij} \\ H^\dagger \otimes Y^\dagger_{ij} & i\gamma^\mu(\partial_\mu + igA_{R\mu}) \otimes 1_{ij} \end{pmatrix} \begin{pmatrix} \psi_{Lj} \\ \psi_{Rj} \end{pmatrix} \tag{19}$$

$$= \begin{pmatrix} \bar{\psi}_{L1} & \bar{\psi}_{L2} & \bar{\psi}_{L3} & \bar{\psi}_{R1} & \bar{\psi}_{R2} & \bar{\psi}_{R3} \end{pmatrix} \begin{pmatrix} i\not{D}_L & 0 & 0 & Y_{11}H & Y_{12}H & Y_{13}H \\ 0 & i\not{D}_L & 0 & Y_{21}H & Y_{22}H & Y_{23}H \\ 0 & 0 & i\not{D}_L & Y_{31}H & Y_{32}H & Y_{33}H \\ Y^\dagger_{11}H^\dagger & Y^\dagger_{12}H^\dagger & Y^\dagger_{13}H^\dagger & i\not{D}_R & 0 & 0 \\ Y^\dagger_{21}H^\dagger & Y^\dagger_{22}H^\dagger & Y^\dagger_{23}H^\dagger & 0 & i\not{D}_R & 0 \\ Y^\dagger_{31}H^\dagger & Y^\dagger_{32}H^\dagger & Y^\dagger_{33}H^\dagger & 0 & 0 & i\not{D}_R \end{pmatrix} \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \\ \psi_{L3} \\ \psi_{R1} \\ \psi_{R2} \\ \psi_{R3} \end{pmatrix}. \tag{20}$$

Here, $Y_{ij} = k L_L^\dagger L_R$. Therefore, Eq (20) shows that the nontrivial flavor structure is induced retaining the gauge interactions are universal, proportional to the identity matrix.

We can express an arbitrary matrix $L \in$ SL(3,C) by the singular value decomposition $L = U\Lambda V^\dagger$. Here, U, V are 3×3 unitary matrices and Λ is the eigenvalue matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1\lambda_2\lambda_3 = 1$. Then, the Yukawa interactions can be described as

$$Y = V_L \Lambda_L U_L^\dagger k U_R \Lambda_R V_R^\dagger, \tag{21}$$

where k is return to a usual coefficient. Redefining the fermions $\psi'_{(L,R)} = V^{\dagger}_{(L,R)}\psi_{(L,R)}$, the final form of the Yukawa matrix found to be

$$Y = \Lambda_L U_L^{\dagger} k U_R \Lambda_R. \tag{22}$$

In order to coincide the phenomenological Yukawa interactions, we require

$$\lambda_{(L,R)3} \gg \lambda_{(L,R)2} \gg \lambda_{(L,R)1}, \quad \lambda_{L3} k \lambda_{R3} \sim 1. \tag{23}$$

Due to $\lambda_1\lambda_2\lambda_3 = 1$, the first condition can be reduced to

$$\lambda_{(L,R)3} \gg 1, \quad \lambda_{(L,R)2} \gg \lambda_{(L,R)1}. \tag{24}$$

It is intriguing that the Yukawa matrix (22) is a restricted class of the seesaw like formula of the partial compositeness [10] in the composite Higgs model [11]:

$$y_{u,d} = \lambda_q M_Q^{-1} Y^{U,D} M_{U,D}^{-1} \lambda_{u,d}, \tag{25}$$

where $\lambda_{q,u,d}, Y^{U,D}$ are arbitrary matrices, $M_{Q,U}$ are diagonal matrices. The Eq. (25) agree with Eq. (22) by setting $Y^{U,D} = k, M_{Q,U,D} = 1$. Then, we can use the same formulation to the composite Higgs model in analyzing the Yukawa matrices. Inversely, it might be possible to divert the idea of SL(3,C) transformation to the partial compositeness.

The origin of the matrices L are obscure. If the SL(3,C) transformation of fermions are also introduced as follows,

$$\psi'_L = L_L^{-1}\psi_L, \quad \bar{\psi}'_L = \bar{\psi}_L L_L, \tag{26}$$
$$\psi'_R = L_R^{-1}\psi_R, \quad \bar{\psi}'_R = \bar{\psi}_R L_R, \tag{27}$$

it leads to a trivial Yukawa interactions and the Lagrangian as before. This fact suggests the existence of an internal noncompact symmetry SL(3,C)$_L\times$ SL(3,C)$_R$ which is broken in some manner. The nature already has the noncompact Lorentz symmetry SO(1,3) $\sim$ SL(2,C), it is not peculiar to assume this kind of entity. In this picture, the matrices L correspond to the "Lorentz transformation" of the internal symmetry, and then, the hierarchy and mixing of flavor are interpreted as the "Lorentz boost" and the "rotation" respectively. Further study will clarify these complications.

4 Conclusions

In this paper, we have proposed a mechanism which induces nontrivial flavor structure from transformations of a noncompact Lie group SL(3,C) in NCG. Matrices $L \in$ SL(3,C) are associated with the preon fields as $a_{L,R}(x) \to L_{L,R}a_{L,R}(x)$. In order to retain the Hermiticity of the Lagrangian, we assume the same trick when $\psi^{\dagger}\psi$ is replaced by $\bar{\psi}\psi$ to construct a Lorentz invariant Lagrangian. As a result, the Dirac Lagrangian has both of flavor-universal gauge interactions and nontrivial Yukawa interactions. Removing the unphysical unitary transformations, Yukawa matrices found to be $Y_{ij} = L_L^{\dagger} k L_R \to$

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